VECTOR POWER IN ALTERNATING-CURRENT CIRCUITS

BY A. E. KENNELLY

It has long been known that in any simple alternating-current circuit, the current and voltage may be conveniently regarded as rotatable vector quantities. It is also known that the power in such circuits is not to be regarded as the vector product of the rotating vector voltage and rotating vector current. It does not seem to have been pointed out, however, that, under certain restrictions, it is proper to regard the power in an alternating-current circuit as a non-rotating vector quantity. Moreover, it does not appear to be generally known, although the fact has not escaped notice, that the imaginary component of vector power, or so-called "wattless power" is, in a restricted sense, just as much power, and just as "wattful" as the real component.

The objects of this paper are:

1. To indicate the limitations under which power in an alternating-current circuit may be treated as a stationary vector;
2. To extend the technology of vector alternating-current quantities;
3. To combat the use of the terms "wattless power" and

"wattless current", offering more logical terms as substitutes, and

4. To offer a plea for the standardization of the direction of phase rotation in the vectors used in alternating-current theory.

Preliminary Definitions. The vectors, or directed magnitudes, employed in alternating-current technology, with rare exceptions that do not come within the scope of this discussion, are all confined to a single plane of reference. That is to say, they relate to two dimensions of space, as distinguished from the vectors of three-dimensional geometry. This limitation may be expressed by saying that the vectors of alternating-current technology are plane vectors. A plane vector may be defined as a quantity having both a direction and a magnitude, but confined to one plane of reference. In what follows, we may for brevity conveniently assume that the term "vector" is an abbreviation for the more strictly logical term "plane vector".

Subdivision of Vectors. There are three classes of vector used in dealing with alternating-current circuits, namely:

1. Vectors that are capable of rotation in their reference plane about a fixed point, and whose projections on a reference axis, or whose intercepts with polar curves, measure the instantaneous values of the quantities represented by the vectors. That is, the rotating vectors may be either projected or intercepted. These vectors may be called rotative vectors.

2. Vectors that are not capable of rotation in their reference plane for any purpose of projective or interceptive representation. These may be called non-rotative vectors.

3. Rotative-vectors that for special purposes are arrested, or treated as though non-rotative. These may be called stationary vectors. Stationary vectors are rotative, but non-rotating.

The above classification and nomenclature may be revealed more clearly by the following table:

<table>
<thead>
<tr>
<th>Nomenclature and algebraic classification of alternating-current vectors.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rotative ((l / \omega t)) { ........................................ Rotating ((l / \omega t))</td>
</tr>
<tr>
<td>{ Stationary ((l / \omega T)) } (T = \text{constant}) } Non-rotating ((l / \theta))</td>
</tr>
</tbody>
</table>
Example of a Rotative Vector. As an example of class (1), let us consider the vector $O E$, Fig. 1, rotating in the plane $O X Y$, about the origin $O$, with a uniform angular velocity of $\omega$ radians per second. Then the length $O E$ may represent to an assigned scale of volts per cm., the maximum cyclic value of a certain sinusoidal e.m.f., say the e.m.f. generated harmonically in the secondary winding of a particular transformer. The direction of rotation may be positive, as indicated by the orbital arrows.
If the frequency of the voltage in the transformer is \( n \) cycles per second; then we know that the angular velocity of rotation must be \( \omega = 2\pi n \) radians per second. Moreover, the vector \( OE \) must, of course, coincide with \( OX \) when the generated voltage attains its maximum positive value.

The circular orbital motion of the end \( E \) of the rotation vector will then correspond to the vector equation:

\[
OE = E_0 \, e^{j\omega t} \quad \text{volt-scale cm. /} \quad (1)
\]

where \( t \) is the time, in seconds, from the start at a particular axis such as \( OX \), \( \varepsilon \) is 2.71828 \ldots, the base of Naperian logarithms, \( j = \sqrt{-1} \), and \( E_0 \) is the maximum cyclic value of the e.m.f. to a voltage scale of length. Equation (1) is sometimes written:

\[
OE = E_0 \, cis \, (\omega t) \quad \text{volt-scale cm. /} \quad (2)
\]

We shall represent either of the above expressions by the briefer and more convenient notation:

\[
OE = E_0 \, /\omega t \quad \text{volt-scale cm. /} \quad (3)
\]

where the quantity \( \omega t \) within the angle sign / means that the angular distance of the radius vector \( OE \) from the initial reference axis is \( \omega t \) radians, or degrees, according to the unit of angle adopted.

As is well known, the orthogonal projection of the radius vector \( OE \) upon the plane of reference \( PP', PP' \), which is parallel to the plane \( OX \), performs a simple harmonic motion. If the reference axis of starting, at time \( t = 0 \), is \( O - Y \), then at any instant, \( t \) seconds thereafter, the distance \( ye \), or projection of \( OE \), will be:

\[
\bar{ye} = E_0 \sin \omega t = E_0 \sin \alpha \quad \text{volt-scale cm.} \quad (4)
\]

which will correspond to the e.m.f. generated at that instant in the transformer.

Again, if the reference axis of starting, at time \( t = 0 \), is \( OX \), then at any time \( t \):

\[
\bar{ye} = E_0 \cos \omega t = E_0 \cos \alpha \quad \text{volt-scale cm.} \quad (5)
\]

If the plane of projection \( PP'P' \) be moved parallel to itself in the direction shown by the arrow at \( \sigma \); or, if with the plane
of projection at rest, the coördinate system of axes moves in the direction $OZ$, with uniform linear velocity, then, as is well known, the projecting point will trace out a sinusoid $o\ e$ having amplitude ordinates $o\ x$, and time abscissas $o\ y$.

**Relation of Phase to the Direction of Rotation**

*Convention No. 1.* If we employ two co-frequent rotative vectors, such as $OE$ and $OI$, Fig. 2, representing say an impressed e.m.f., to a certain volt-scale, in a simple alternating-current circuit, and the current strength thereby produced in the same circuit, to a certain ampere-scale; then, if there is inductive reactance in the circuit, we know that the current will lag behind the impressed e.m.f. This means that the orthogonal projection of $OI$ on the axis $OX$ of reference, must reach its maximum after $OE$ has passed its maximum projection on that
axis. Consequently, with the plane $XOZ$, fixed in space, and with the vectors $OE$, $OI$, rotating in the positive direction, as shown by the orbital arrows, $OI$ must make a negative angle with $OE$; or, $OE$ must make a positive angle with $OI$; so that the angle $EOI$ is trigonometrically negative, as shown in Fig. 2. If the vector $OE$ starts from the position $OX$, at time $t = 0$, the orthogonal projection of $OI$ proceeds to execute on the axis $OX$ the cosinusoid:

$$\overline{Oi} = I_0 \cos (\omega t - \theta)$$

ampere-scale cm. (6)

Fig. 3.—Pair of rotative vectors of the same frequency and their cosinusoidal projections. Representation inverse. Current lagging. Isometric projection. $XOY$ plane of rotation. $XOZ$ plane of rotation.

If the origin $O$ and system of axes advance uniformly with respect to the stationary plane of projection $XOZ$ in the direction $OZ$; or, what is equivalent, if the plane of projection moves in the direction $OZ$ past a fixed origin $O$, the cosinusoids $e'e$ and $i'i$ will be traced as curves with amplitude ordinates, and times as abscissas. This convention was adopted in alternating-current technology by Fleming in 1887.¹

¹. Fleming, loc. cit.
Convention No. 2. If, however, we reverse the direction of rotation of the vectors $OE, OI$ in their orbit; i.e., if we adopt negative rotation, we shall require that the leading vector make a negative angle with the lagging vector, a condition opposite to that above defined. Or, keeping the direction of rotation positive, we may assume that the vectors are fixed in their plane; but that the plane of projection rotates positively at the uniform angular velocity $\omega$. Thus in Fig. 3, the two vectors $OE$ and $OI$ may respectively represent, as before, the impressed e.m.f. and the current in an inductively reactive circuit. If these two vectors remain fixed; but the axis $-XOX$, and with it the plane of projection $ZOX$, rotates in the positive direction, of the curved arrows, about the $OZ$ axis, the e.m.f. vector $OE$ must make a negative angle with the lagging current vector $OI$; or the pair will take relatively opposite positions to those they had in Fig. 2, and the angle $EOI$ will be positive.

Convention No. 3. Some writers, instead of employing rotative vectors to be projected orthogonally upon an axis of reference, prefer to represent simple harmonic motion by the device of an intercepting circle. Thus, in Fig. 4, the heavy circle $OGEF$, in the plane $XOY$, may be regarded as stationary in space, and the axis $OR$, sometimes called a "time-axis," rotates positively in this plane about the origin $O$, as shown by the curved arrows, with uniform angular velocity $\omega$ radians per second, or $n$ revolutions per second, commencing at the position $O-Y$, when $t=0$. As the axis $OR$ advances, it becomes intercepted by the circle $OGE$, and forms to that circle a chord of increasing length, until it reaches the position occupied in the Figure by $-XOX$, when the chord will have become a diameter, and the length of the moving axis intercepted by the fixed circle will be a maximum in the positive direction. As the rotation of $OR$ about $O$ continues, the length intercepted by the circle will diminish, until it will be zero in the position $OY$. Continuing the rotation, we may adopt either of two equivalent conventions, between which writers are divided. We may either use a second circle, shown in dotted lines at $Ogef$, equal and opposite to the first, and consider this to be a negative circle, such that all intercepts upon

3. Kapp, loc. cit., Fig. 8.
$OR$ shall be considered negative, intercepts on $Or$ being ignored; or, we may dispense with the second circle, and allow intercepts on $Or$ to count as negative intercepts, during the second half of the revolution. In either case, the length of the intercept on the moving axis will follow a simple harmonic law, according to the expression $OE \sin \omega t$ units of length, $OE$ being the diameter of the intercepting circle.

If the rotating axis starts, at time $t = 0$, from the position $-XOX$ in Fig. 4, the length intercepted by the fixed circle will likewise follow a simple harmonic law according to the expression $OE \cos \omega t$.

Fig. 4.—Uniformly rotating vector and fixed intercepting circle, with simple harmonic intercept. Isometric projection. $XOY$ plane of rotation.

In order to represent the relation between a simple harmonic current in an inductively reactive circuit under a simple harmonic e.m.f. by convention 3, two intercepting circles are required, as in Fig. 5. Here the negative counterparts are omitted, and negative intercepts are assumed with each circle. As shown in the figure, it is necessary, with the positive direction of rotation of the moving axis $OR$, to have the diameter of the circle $OI$ make a positive angle with the diameter of the circle $OE$, in order that the intercept on the current circle shall reach its maximum later than the intercept on the voltage circle. A lagging current, therefore, requires a positive angle $EOI$. 

[Diagram of uniformly rotating vector and fixed intercepting circle, with simple harmonic intercept. Isometric projection. $XOY$ plane of rotation.]
In the practical use of convention 3, the circles are commonly omitted, for convenience, and are merely represented by their diameters $OE$ and $OI$, which become respectively the stationary-vector e.m.f. and current of the diagram.

As regards the use or disuse of the negative circle, as in Fig. 4, it is simpler to dispense with it, and to use negative intercepts on $OY$, except when the positive and negative waves are not symmetrical. In that case, the retention of the negative loop simplifies the diagram, since it avoids superposition of loops, and ambiguity of paths.

![Diagram](image)

**Fig. 5.—Pair of intercepting circles representing a simple harmonic e.m.f. and a simple harmonic current lagging 90 deg. behind the same. Isometric projection. $XOY$ plane of rotation. Representation inverse.**

**Convention No. 4.** If we assume that the axes of Fig. 5 are fixed in space, but that the intercepting circles $OE$ and $OI$ rotate together in the positive direction about the center $O$, measuring off from moment to moment intercepts on some axis, say $OX$, equal to the respective instantaneous voltage and current; then in order to represent an e.m.f. and a lagging current, it will be necessary for the diameter of $OE$ to make a positive angle $IOE$ with the diameter of $OI$, or the angle $EOI$ will be negative, the opposite condition to that in Fig. 5. According
then to convention 4, Fig. 5 would represent a current $OI$ leading the e.m.f. $OE$ by 90 degrees.¹

Convention No. 4 does not seem to be in use, but is introduced here in order to complete the classification symmetrically, see Table II.

**TABLE II**

Conventions Employed in Rotative - Vector Diagrams

<table>
<thead>
<tr>
<th>Convention</th>
<th>Direction of rotation</th>
<th>Plane of projection</th>
<th>Vectors</th>
<th>Intercepting circles</th>
<th>Vector</th>
<th>Angle of $EOI$ for an inductive circuit</th>
<th>Use</th>
<th>Date of use</th>
<th>Expression of inductive impedance</th>
<th>Representation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type No.</td>
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<tr>
<td>1</td>
<td>+ Fixed</td>
<td>Rotating</td>
<td></td>
<td></td>
<td>-</td>
<td>Fleming Blakesley</td>
<td>1887</td>
<td>1889</td>
<td>$r+ix$</td>
<td>Direct</td>
</tr>
<tr>
<td></td>
<td>- Rotating</td>
<td>Fixed</td>
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<td>-</td>
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<tr>
<td></td>
<td>+ Rotating</td>
<td>Fixed</td>
<td></td>
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<td>+</td>
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<td>Inverse</td>
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<tr>
<td></td>
<td>- Fixed</td>
<td>Rotating</td>
<td></td>
<td></td>
<td>+</td>
<td>Kapp</td>
<td>1889</td>
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<td>2</td>
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<tr>
<td>3</td>
<td>+</td>
<td>Fixed</td>
<td>Rotating</td>
<td></td>
<td>+</td>
<td>Kapp Steinmetz</td>
<td>1889</td>
<td>1893</td>
<td>$r-ix$</td>
<td>Direct</td>
</tr>
<tr>
<td></td>
<td>-</td>
<td>Rotating</td>
<td>Fixed</td>
<td></td>
<td>-</td>
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<td></td>
</tr>
<tr>
<td>4</td>
<td>+</td>
<td>Rotating</td>
<td>Fixed</td>
<td></td>
<td>-</td>
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<td>Inverse</td>
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<tr>
<td></td>
<td>-</td>
<td>Fixed</td>
<td>Rotating</td>
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</table>

**Representation of Complex Harmonic Quantities**

It has been claimed that a complex harmonic quantity is only capable of being represented as a closed curve to polar coordinates by the method of the intercepting curve, and that the projecting curve, or "clock-diagram", cannot be used in

¹ The above four conventions by no means exhaust the possibilities of rotative-vector representation. For example, as a subtype of interceptive or polar representation, we might assume the two circles $OE, OI$ of Fig. 5 to have their diameters on one and the same line, instead of being angularly displaced. Two angularly rotating displaced vectors could then be employed for e.m.f. and current intercepts respectively, instead of the single rotating vector $OR$. With such an arrangement, a lagging vector current $OI$ would make a negative angle with the e.m.f. vector $OE,$ or would reverse the relations of Fig. 5. That is, it would produce direct representation. Since, however, this method does not seem to have been used, it is omitted from the Classified Table of Conventions.
such cases.¹ It is true that the simple projecting-circle or clock-diagram, with uniform angular velocity of the radius vector, cannot represent a non-sinusoidal or complex harmonic wave; just as it is true that the simple intercepting circle, with uniform angular velocity of the radius vector, cannot represent a complex harmonic wave. But in the same manner that a change in the

![Diagram](image)

**Fig. 6.—Sinusoidal wave and its projecting circle and its intercepting circle.**

form of intercepting curve will permit of any complex wave being presented in polar coordinates, so a corresponding change in the form of the projecting curve will permit of the same result. Thus in Fig. 6, the sinusoid $A B C D$, drawn to rectangular coordinates, may be represented either by the projecting circle

---

a, b, c, d, or by the intercepting circle \( a, b, c, d \), with or without its dotted neighbor. Similarly, in Fig. 7, the triangular wave \( A B C D \), drawn to rectangular co-ordinates, may be represented either by the projecting curve \( a, b, c, d \), or by the intercepting curve \( a, b, r, O \), with or without its dotted neighbor. The relation between corresponding radii on the polar curves is always:

\[
\overline{OR} = \overline{OR} \cos \theta \quad \text{cm. (7)}
\]

![Diagram](image)

**Fig. 7.**—Triangular wave, its projecting curve, and its intercepting curve.

where \( \overline{OR} \) is the radius vector (Figs. 6 and 7) of the intersecting curve, at the angle \( \theta \) with the initial line of reference, and \( \overline{OR} \) is the corresponding radius vector of the projecting curve. Consequently, having given either the polar projecting curve, or the polar intercepting curve of any complex harmonic alternating-current wave, the other can be immediately deduced.

It will be evident from the above considerations that with a pair of rotative vectors (Fig. 8) in the plane \( XOY \), one \( OE \)
representing an impressed e.m.f., and the other $OI$ the resulting current, in a simple alternating-current circuit, the question as to whether $OI$ is to be interpreted as a leading or a lagging current does not depend upon the use of projecting as against intercepting curves. It may be either a leading or a lagging current with either the "clock" diagram (Fig. 1) or the "spectacles" diagram (Fig. 4). It depends entirely upon the convention employed. If, with a projecting curve, the two vectors of Fig. 8 rotate (in the positive direction) as in convention No. 1, then $OI$ represents a lagging current. If on the contrary, the vectors are to be considered as stationary, and the axis of refer-

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{vector_power.png}
\caption{Vector e.m.f. and current in simple alternating-current circuit. The current is either a leading or a lagging current according to which convention used in vector representation. Isometric projection. $XOY$ plane of rotation.}
\end{figure}

ence rotates positively with respect to them, as in convention No. 2, then $OI$ represents a leading current. Again, if with a fixed intersecting axis $OX$, a pair of circles located with their rotating diameters on $OE$ and $OI$, the vectors of Fig. 8 rotate in the positive direction, as in convention No. 4, $OI$ represents a lagging current. Finally, if the vectors $OE$ and $OI$ are fixed, and represent the diameters of fixed intersecting circles, with respect to which an axis of reference rotates positively, as in convention No. 3, $OI$ represents a leading current.

Aside from mental habit and psychological inertia, any one of these conventions appears to be as good as another. If the positive direction of rotation is understood in all cases; each con-
vention appears to be logical and systematic. It cannot be maintained that one or more are right and the rest are wrong. The question is essentially one of arbitrary convention, not of demonstration. Nevertheless, it is very important that the matter should be settled definitely by general agreement. Much ambiguity results from the present dissension; because it is often difficult to ascertain, when opening a text-book in order to consult it, which method of representation the author follows. Imagine the confusion which would ensue in the worlds of pure and applied mathematics, if it were left to the choice of each writer to select which direction of the axis — $X O X$ Fig. 8, say, should be positive; so that some books should be written on the assumption of $O X$, Fig. 8, being plus, and others on the basis of $O X$ being minus. Or again, suppose it were left open to individual selection, which direction of rotation in a plane, clockwise, or counter-clockwise, should be taken as positive. These arbitrary selections have long been fixed by universal agreement among mathematicians. Yet this is the kind of dissension which exists to-day in the world of vector alternating-current technology.

Moreover, it is not enough that the decision should be made and accepted nationally. The only satisfactory decision must be made and accepted internationally.

**Direct and Inverse Representation**

In what follows, this paper will conform to what seems the majority of opinion on this matter, and $O I$ in Figs. 8 and 10, will be regarded as a lagging current with respect to the e.m.f. $O E$. This means adhering to conventions 1 and 4, and by preference to convention 1. This method of representation which, in the direction of positive rotation, makes a *leading* current *lead*, and a *lagging* current *lag*, with respect to its e.m.f., will be called, for the purposes of distinction, *direct representation*, and the opposite method, involving either convention 2 or convention 3, will be called *inverse representation*.

Tables III and IV contain lists, which are by no means exhaustive, of publications using direct and inverse representations.

1. The urgent need for the standardization of alternating-current vector rotation has been pointed out by various writers, both in this country and in Europe. See W. S. Franklin: A discussion of some points in Alternating-Current Theory. Transactions A.I.E.E., May 1903, Vol. 21, pp. 589–601, and Carl Richter, Alternating-current Diagrams, Elek. u. Maschinenbau, July 12, 1908, pp. 608, 609.
respectively. No attempt has been made to discover more publications using one method than the other, and in the search that led to the formulation of these lists, no publications which contained vector diagrams were discarded, except such as made it difficult to decide which method was followed. The fact that 42 publications appear in the list of direct representation and 24 in the list of inverse representation, indicates a distinct preponderance in favor of direct representation. The dissension is not confined to any single country, or group of countries, and it dates as far back as 1889 at least.

**TABLE III**

**Publications Employing Direct Representation of Alternating-Current Vectors**

<table>
<thead>
<tr>
<th>Name of publication</th>
<th>Author</th>
<th>Publisher</th>
<th>Date</th>
</tr>
</thead>
<tbody>
<tr>
<td>2. Electrical Engineer's Pocket Book</td>
<td>H. A. Foster</td>
<td></td>
<td>1908</td>
</tr>
<tr>
<td>3. Alternating-Current Machines.</td>
<td>Sheldon, Mason and Hausmann</td>
<td></td>
<td>1909</td>
</tr>
<tr>
<td>5. Munroe &amp; Jamieson's Pocket Book</td>
<td></td>
<td>Ch. Griffin &amp; Co., London</td>
<td>1908</td>
</tr>
<tr>
<td>7. Elements of Electrical Engineering</td>
<td>Franklin &amp; Esty</td>
<td>MacMillan Co.</td>
<td>1909</td>
</tr>
<tr>
<td>8. Electric Waves</td>
<td>W. S. Franklin</td>
<td></td>
<td>1909</td>
</tr>
<tr>
<td>11. Electric Power Transmission</td>
<td>L. Bell</td>
<td></td>
<td>1907</td>
</tr>
<tr>
<td>12. Problems in Electrical Engineering</td>
<td>W. V. Lyon</td>
<td></td>
<td>1908</td>
</tr>
<tr>
<td>13. Electrical Engineering Leaflets</td>
<td>Houston &amp; Kennelly</td>
<td>The Elec. Engineer</td>
<td>1897</td>
</tr>
<tr>
<td>15. Alternating Currents</td>
<td>Bedell &amp; Crehore</td>
<td>W. J. Johnston Co.</td>
<td>1893</td>
</tr>
<tr>
<td>19. The Elements of Alternating Currents</td>
<td>Franklin &amp; William- son</td>
<td>MacMillan Co.</td>
<td>1901</td>
</tr>
<tr>
<td>21. Electrical Measurements</td>
<td>Carhart &amp; Patterson</td>
<td>Allyn &amp; Bacon</td>
<td>1895</td>
</tr>
<tr>
<td>22. A Text Book of Electrical Machinery</td>
<td>Ryan, Norris and Hoxie</td>
<td>John Wiley</td>
<td>1903</td>
</tr>
<tr>
<td>23. The Principles of A. C. Working</td>
<td>A. Hay</td>
<td>Biggs &amp; Co.</td>
<td>1897</td>
</tr>
<tr>
<td>24. Telephone Lines and their Properties</td>
<td>W. J. Hopkins</td>
<td>Longmans Green</td>
<td>1894</td>
</tr>
<tr>
<td>25. Experimental Electrical Engineering</td>
<td>V. Karapetoff</td>
<td>John Wiley</td>
<td>1908</td>
</tr>
<tr>
<td>26. Electrical Problems</td>
<td>Hooper &amp; Wells</td>
<td>Ginn &amp; Co.</td>
<td>1902</td>
</tr>
<tr>
<td>27. The Dynamo</td>
<td>Hawkins &amp; Wallis</td>
<td>MacMillan Co.</td>
<td>1909</td>
</tr>
<tr>
<td>28. Electricity and Magnetism</td>
<td>F. E. Nipher</td>
<td>J. L. Boland Co.</td>
<td>1885</td>
</tr>
</tbody>
</table>
TABLE IV

Publications Employing Inverse Representation of Alternating-Current Vectors

<table>
<thead>
<tr>
<th>Name of Publication</th>
<th>Author(s)</th>
<th>Publisher</th>
<th>Date</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dynamo-Electric Machinery</td>
<td>S. P. Thompson</td>
<td>Spon &amp; Chamberlain</td>
<td>1905</td>
</tr>
<tr>
<td>Elements of Electrical Engineering</td>
<td>C. P. Steinmetz</td>
<td>El. W. &amp; Engr.</td>
<td>1902</td>
</tr>
<tr>
<td>Whittaker’s El. Engr’s. Pocket Book</td>
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<td>Whittaker &amp; Co.</td>
<td>1906</td>
</tr>
<tr>
<td>Electrical Engineering</td>
<td>Rosenberg, Gee &amp; Kinzbrunner</td>
<td>Longmans Green</td>
<td>1907</td>
</tr>
<tr>
<td>The Induction Motor</td>
<td>B. A. Behrend</td>
<td>El. W. &amp; Engr.</td>
<td>1901</td>
</tr>
<tr>
<td>Transformers</td>
<td>Gisbert Kapp</td>
<td>Whittaker &amp; Co.</td>
<td>1908</td>
</tr>
<tr>
<td>Vectors and Vector Diagrams</td>
<td>Cramp &amp; Smith</td>
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<td>Alternating-Current Engineering</td>
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<td>A. Still</td>
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Example of a Non-rotative Vector

A simple example of a non-rotative vector is an ordinary impedance of the type $z/\theta$ ohms, as indicated in Fig. 9. An ordinary impedance does not pass through zero cyclically, like an alternating e.m.f. or current, and no practical use is at present derivable from the notion of rotating a vector impedance about its origin. In direct representation, the impedance of Fig. 9 is essentially an inductive impedance, as distinguished from a condensive impedance. That is, it represents the impedance of some particular reactance coil. With inverse representation, however, it would necessarily represent a condensive impedance, or the impedance of some particular condenser, operated at a certain frequency, in series with a certain resistance.

Impedances, admittances, reluctances and permeances, when treated as vectors, are essentially non-rotative vectors.

Quantitatively, the application of a non-rotative vector to a rotative vector as a multiplying factor alters both the magnitude and the phase of the resultant rotative vector, without altering the frequency or angular velocity of rotation. Thus the relation:

$$I/\omega t \times Z/\theta = I Z/\omega t + \theta \text{ volts} \tag{8}$$

indicates that the product of a vector current $I$ amperes, rotating with the angular velocity $\omega$ radians per second, and a vector impedance of $Z$ ohms, with the fixed angle $\theta$ radians, gives rise to a vector voltage $I Z$, rotating with the same angular velocity as $I$, but advanced $\theta$ radians in phase beyond $I$. The orthogonal projection of the voltage product will follow a simple harmonic motion, $\theta$ radians advanced in phase with respect to the corresponding motion of $I$.

Degradation of a Rotative Vector into a Stationary Vector

It frequently happens that a quantity which is capable of being regarded as a rotative-vector quantity, needs only to be designated as a vector in respect of phase relation to another vector or vectors; as, for example, when a simple harmonic alternating current, of say 10 amperes r.m.s., is desired to be designated as a vector lagging, perhaps 37 degrees in phase, behind a simple
harmonic impressed e.m.f. of 100 volts r.m.s. It would be possible to give a direct representation of this condition as in Fig 2 with a rotative vector $OE$ of 141.4 volt-scale length, followed at 37 deg. by a rotative vector $OI$ of 14.14 ampere-scale length. But if there is no necessity for drawing attention to the orthogonally projective properties of these vectors, they may be represented as a simple non-rotative pair, $OE$, $OI$, Fig. 10, in which case it is convenient to use their virtual or root-mean-square values, instead of their maximum values. We may consider that, quantitatively, this diagram represents the following conditions:

$$OE = OI/0^\circ \times Z/37^\circ \quad \text{volt-scale cm.} / \_ (9)$$

and

$$OI = OE/0^\circ \pm Z/37^\circ \quad \text{ampere-scale cm.} / \_ (10)$$

That is, there is some non-rotative vector impedance of $10/37^\circ$ ohms which connects, by Ohm's law, a virtual e.m.f. of 100 volts with a virtual current of 10 amperes lagging 37 deg. behind it.

In general, therefore, a non-rotative vector operator, such as an impedance, multiplied into a rotative vector, produces a rotative vector of changed amplitude and phase, but when multiplied either into a pure non-rotative vector, or into a stationary vector, produces a non-rotating vector of changed amplitude and phase, according to the property of multiplication of complex numbers:

$$a/\alpha \times b/\beta = a/b/\alpha + \beta \quad \text{numeric} / \_ (11)$$

**Power in Simple Alternating-Current Circuits**

It is well known that if a simple harmonic e.m.f. $E_\theta \cos \omega t$ volts, where $E_\theta$ is the maximum cyclic value, propels a simple harmonic current $I_\theta \cos (\omega t + \theta)$ amperes; so that the e.m.f. and current differ in phase by the positive or negative angle $\theta$, the electric power developed in the circuit by the source of e.m.f. on the current is at any instant:

$$P_t = \frac{E_\theta I_\theta}{2} \left\{ \cos \theta + \cos (2\omega t + \theta) \right\} \quad \text{watts} (12)$$

$$= EI \left\{ \cos \theta + \cos (2\omega t + \theta) \right\} \quad \_ (13)$$
where $E$ and $I$ are respectively the virtual or root-mean-square e.m.f. and current; or if $P$ be the product $EI$, of root-mean-square volts and amperes:

$$p_t = P \{\cos \theta + \cos (2\omega t + \theta)\} \text{ watts (14)}$$

Consequently, any projective or interceptive rotating vector which represents the power in a simple alternating-current circuit must possess an angular velocity double that of the e.m.f. or current. Moreover the origin or axis of the rotating vector power must be displaced from the origin of the e.m.f. and current components.

It has been pointed out by Mr. J. Irving Brewer\(^1\) that a construction which satisfies the rotative-vector power relations is to lay off $OE$, the maximum cyclic e.m.f., along the $OX$ axis, (Fig. 11) and $OI$ at the proper phase angle; in the case presented,

Then on $OP$ at a point $P'$, such that $OP' = EI$, to watt-scale, is the center of the rotating power vector which starts with doubled angular velocity from the position $P'P$, at the moment when $E_0$ starts from $OX$, and $I_0$ from $OP$.

Under these conditions, the three rotating vectors will project on the fixed axis $-XOX$, and on the moving plane $XOZ$, the respective instantaneous values of power, e.m.f. and current.

**Non-rotative Vector Power**

If a simple harmonic current of $I$ root-mean-square amperes flows through an alternating-current circuit of impedance $R + jX = Z/\alpha$ ohms, $(AC$, Fig. 12), the root-mean-square potential-difference $E$ volts on the circuit will be obtained by the operation:

$$E/\alpha = I/0 \times Z/\alpha = IZ/\alpha \text{ root-mean-square volts /}$$

Here $I$ is taken either as an ordinary real number, or as a plane vector number of zero angle, and therefore at standard phase. That is, $I$ is taken as a stationary vector, so that $E$ is also a stationary vector e.m.f., $(DF$ Fig. 12), advanced in phase $\alpha$ radians or degrees ahead of the current. The real component $DE$, is the effective root-mean-square component of potential difference, so far as concerns the average liberation of power from the source into the circuit, and the imaginary component $EF = jI X$, is the reactive root-mean-square component potential-difference, or the component which develops reactive power on the current. This reactive power is directed from source to circuit, and back again, in one power cycle, or in one half-cycle of current.

Again, if we multiply the non-rotative vector p.d. $DF$ by the stationary, vector current, according to the formula:

$$P/\alpha = I/0 \times E/\alpha = I E/\alpha = E I/\alpha \text{ watts /}$$

we obtain a stationary vector power $GK$, advanced $\alpha$ degrees or radians ahead of the current. This power is the apparent power, commonly called volt-amperes. It is perhaps practically advantageous to call the unit of apparent power the volt-ampere in order to distinguish apparent power from effective power in engineering; but a volt-ampere is essentially a watt, and the apparent power is correctly stated as apparent or resultant watts,
the vector sum of effective and reactive watts. The effective component $GH$ is the average power delivered to the circuit by the generator, and is usually called the "real power". The reactive power $HK$ is, however, when considered from within the circuit, just as real as the effective power $GH$; so that the term "real power" is unsuitable. The reactive power $HK$ is the maximum cyclic power expended in transmitting energy into and out of the magnetic flux linked with the circuit, being alternately plus and minus, or from and to the generator, in successive quarter cycles of current. This energy is kept in the circuit; whereas the effective power $HG$ transmits energy out of the circuit. The maximum reactive cyclic power $HK$ is all internal. The effective power $GH$ is the average of that delivered externally, is the cyclic average of the instantaneous total internal power, and is also the maximum cyclic value of the externally delivered power.

Finally, if we divide the stationary vector power by $2\omega$ according to the equation:

$$\frac{W}{\alpha} = \frac{P}{\alpha} / (2\omega) \quad \text{joules per energy cycle}$$  (17)

we obtain the stationary energy vector $LN$. This is the maximum cyclic apparent, or resultant oscillatory, energy delivered by the generator to the circuit in each energy cycle, over and above the average effective energy delivered at the rate $GH$. The perpendicular component $MN$ is the maximum cyclic change of reactive energy in the magnetic flux of the circuit. The horizontal component $LM$ is the maximum cyclic oscillation of effective energy.

A practical example will illustrate the above conditions. We may assume a large single-phase 60-cycle alternator delivering at its switchboard eight megawatts (8,000 kw.) of effective power, and three megawatts of reactive power, or 8.544 apparent megawatts, under a power factor of 0.936. If the delivered current is 4,000 amperes, its terminal voltage will be 2,136 volts.

The stationary-vector power diagram for this generator is shown at $GHK$ in Fig. 12. The analysis of the circuit conditions is given in Fig. 13, to rectangular coördinates and to

1. A useful diagram of this type—essentially a stationary-vector power diagram—appears at page 540 of Mr. Percy H. Thomas' paper on "Calculation of the High-Tension Line". PROCEEDINGS of A.I.E.E., June, 1909.
sinusoidal current phase. The current $I$ has a maximum cyclic strength of $4000\sqrt{2}$, or 5,656 amperes. The potential difference $E$ is ahead of the current by $\alpha = 20.6$ deg. It is analyzed into the effective component ($DE$ Fig. 12) of 2,000 volts root-mean-square, or $E_f = 2,828$ volts maximum (Fig. 13) and the reactive component $EF$ of 750 volts root-mean-square ($E_k = 1061$ volts maximum). The product $IE_f$ produces the effective power $P_f$ of eight megawatts amplitude, ($GH$ Fig. 12) above and below the average $P_m$ of eight megawatts. The product $IE_k$ produces the reactive power $P_k$ of three megawatts.
amplitude \((HK\text{ Fig. 12})\). The sum of these two quadrature power components is the total apparent power \(IE = P\) of 8.544 megawatts above and below the mean \(P_m\).

The reactive power \(P_k\) is associated with a cyclic energy change of \(W_k = 0.3979\) myriajoule = 3,979 joules = 405.6 kilogram-meters or 2940 ft-lb. This energy \((MN\text{ Fig. 15})\) is stored in, or removed from, the magnetic flux of the circuit in one quarter of an energy-cycle, or one eighth of a current-cycle \((0.00208\text{ second})\) over and above a steady stock of energy \(o\ o\) \((\text{Fig. 13})\) of equal amount. The total magnetic energy in a current half-cycle is therefore:

\[
2W_k = I^2 X/\omega \quad \text{joules per cycle of current} \quad (18)
\]
or 7958 joules (0.7958 myriajoule), at 90 deg. and at 270 deg. of current-phase.

The effective power \( P_f \) is associated with a cyclic energy change of \( W_f = 1.061 \) myriajoule in each cycle (L M Fig. 12) above and below the average of eight megajoules per second, or 13.333 myriajoules per current cycle, delivered by the generator outside of the circuit.

![Graph](image)

Fig. 15.—Analysis of current, power and energy to potential-difference as standard of phase.

The total cyclic energy change is \( W = 1.133 \) myriajoule (L N Fig. 12) above and below the average of 6.667 myriajoules per energy cycle.

Fig. 12 contains, therefore, a non-rotative impedance triangle \( A B C \) of ohms, a stationary potential-difference triangle \( D E F \) of root-mean-square volts, a stationary power triangle \( G H K \) of
maximum cyclic or amplitude watts, and a stationary energy
triangle $LMN$ of maximum cyclic or amplitude joules. These
four non-rotating vector triangles pertain to every alternating-
current circuit, considered with reference to the stationary vector
current $I_0$ root-mean-square amperes. If the circuit contains
condensance, instead of inductive reactance, the four triangles
will all be inverted, with negative reactive quantities, or of the
geometrical type indicated in Fig. 14. If the current and p. d.
in the circuit are sinusoidal, then all of the 18 vector quantities
involved will be strictly interpretable by simple harmonic theory.
If the current, or the potential-difference, or both, are only
approximately sinusoidal, the vector triangles may still be com-
puted, and they permit of being considered as "equivalent sinu-
soidal" triangles. In this case, however, the various vector
quantities cease to be strictly interpretable physically. Finally,
if the current or potential-difference, or both, depart widely
from the sinusoidal type, the vector triangles, although still
existing logically and geometrically, may fail completely to be
interpreted physically. That is, the vector impedance, voltage,
power and energy may be inconsistent with the physical con-
ditions. In such cases, it is necessary to analyze the current and
potential-difference into harmonic components, develop a series
of vector triangles, one for each component, and aggregate the
separate effects.

If in a single-frequency (simple harmonic) circuit, the im-
pedance triangle be given and the root-mean-square current, the
other three vector triangles of $E$, $P$ and $W$, follow immediat-
ey, and without any ambiguity. That is, the stationary-vector
series $Z$, $E$, $P$ and $W$ is unique. But if either the power or energy
triangle be given initially, the $Z$ triangle which follows therefrom
is ambiguous, because there may be condensance associated
with the reactance, either in series or parallel, and a doubly in-
finité system of circuits could therefore be devised that would
satisfy the $W$, $P$, and $E$ vectors. The only definite conclusion
in such a case is that the reactance preponderates over the
condensance to the amount indicated by the $X$ of the $Z$ triangle.

If one or more impressed counter-electromotive forces exists in

May, 1894, Vol. 11, pp. 570, 616.
2. Steinmetz, "Symbolic Representation of General Alternating
Waves and of Double-Frequency Vector Products", TRANSACTIONS
the circuit, as, for example, that of a synchronous alternating-current motor, the $Z$ diagram that follows from a given $P$ diagram and current $I$, is logically consistent, and may be practically useful, but is purely fictitious.

If instead of taking the phase of the current as standard, we take the phase of the potential-difference as standard, we exchange $E/0$ for $I/0$ as the fundamental stationary vector, and we obtain the four stationary vectors $Y$, $I$, $P$, and $W$ of Fig. 14, connected by the relations:

$$ I \{ \alpha = E/0 \cdot Y \{ \alpha \quad \text{root-mean-square amperes} / \_ (19) $$

$$ P \{ \alpha = E/0 \cdot I \{ \alpha \quad \text{max. cyclic watts} / \_ (20) $$

$$ W \{ \alpha = (P \{ \alpha )/ (2 \omega) \quad \text{max. cyclic joules} / \_ (21) $$

It will be observed that the $P$ and $W$ triangles in Fig. 14 are inverted by comparison with those in Fig. 12, and yet the same single-phase alternator is supposed to be operating on the same circuit in each case. The anomaly is explained by the fact that as shown by Figs. 13 and 15, the resultant power curve $P$ lies intermediate in phase between the current curve $I$ and the potential-difference curve $E$, so that while the power is leading with respect to the current, it is lagging with respect to the potential-difference. Consequently, the stationary power vector of an alternating-current circuit has, with direct representation, either a negative or a positive angle, according as the phase of one or other of the two quantities $E$ and $I$ is taken as standard, as well as whether the circuit is reactive or condensive.

By comparing Figs. 13 and 15, it will be seen that the curves $E$, $I$, $P$ and $W$, or resultant potential-difference, current, power and energy, all correspond, or may be superposed each on each; but the components $E_j$, $E_k$, $I_j$, $I_k$, $P_j$, $P_k$ and $W_j$, $W_k$ lie on reversed sides of their respective resultants in the two figures, as called for by the two sets of stationary-vector diagrams in Figs. 12 and 14.

It follows therefore that the $P$ and $W$ stationary-vector diagrams exist in two mutually inverted forms for every single-frequency alternating-current circuit; whereas the $E$, $I$, $Z$ and $Y$ stationary-vector diagrams are single, their position (erect or inverted), depending upon whether the circuit is inductively or
condensively reactant, as well as upon whether direct or inverse representation is employed.

**Rotative Properties of the Stationary Vectors $E, I, P$, and $W$**

In Fig. 12 the vector $E = IZ$ is taken as stationary; but it is of course capable also of being considered as a rotative vector, rotating about the vertex $D$, with the angular velocity $\omega$. In that case, the three voltages $E, E_f$ and $E_k$ should each be increased in the ratio of $\sqrt{2}$, or should be changed in scale, in order that their orthogonal projections on a stationary reference axis, or interceptions on a stationary circle, should correspond to the several instantaneous voltages of impressed potential-difference, effective potential-difference and reactive potential-difference.

In Fig. 12 the power vector $P = I^2 Z$ is also taken as stationary; but it is capable of being considered as a rotative vector without any change of scale, by rotating the triangle $GHK$, about the vertex $K$, with the positive angular velocity $2\omega$. This is represented in Fig. 16, where $K$ is the center of rotation.
in the plane $XOY$ of the paper, $Kg = HG$, and the extreme left-hand projection of $g$ is taken at $o$, as the origin of power co ordinates, in the plane $XOZ$ of projection. The projection of $K$, at $P_m$, then marks off the steady average power of the system in watts, while the projection points $PP$, marking the limits of the rotating vector $I^2Z$, indicate the range of cyclic oscillation of the power. Thus, as shown in Fig. 13, the power pulsates between $-0.544$ and $+16.544$ megawatts once in each power-cycle, or half current-cycle. Similarly, the rotating vector $KH$ projects, with respect to the center $P_m$, the instantaneous magnitude of the reactive power $P_k$ about the zero line $OO$, Fig. 13. In accordance with Fig. 13, the rotation of Fig. 16 is assumed to start at a moment when the vector current $I$, rotating with angular velocity, $\omega$, if applied with its center on $K$, would occupy the direction $KH$.

Similarly, if we rotate the power triangle in Fig. 14 about its vertex $k$, with the angular velocity $2\omega$, we shall obtain the same diagram as Fig. 16, except that the phase order of advance will be reversed; i.e., $Pf$, followed by $P$ and $Pk$, instead of $Pk$ followed by $P$, and $Pf$. The orthogonal projections of these three vectors will then correspond to the curves $Pf$, $P$ and $Pk$ as given in Fig. 15 to potential-difference phase.

In Fig. 12, the energy vector $W = I^2Z/(2\omega)$ is also taken as stationary; but it is capable of being considered as a rotative vector without any change of scale, by rotating the triangle $LMN$ about the vertex $N$ with the positive angular velocity $2\omega$, and taking instantaneous projections—not on the $XX$ axis—but on the $YY$ axis, as shown in Fig. 17. Here the successive rotating vectors, $W_k$, $W$, and $W_f$ project the sinusoidal curves $W_k$, $W$, and $W_f$ of Fig. 13, each with respect to its own zero line. At the moment represented in Fig. 17, the current $I$ is supposed to start with angular velocity $\omega$ from the position $NM$, with center $N$, and to project orthogonally on the $XX$ axis. Similarly, if we rotate the stationary energy triangle $lmn$ of Fig. 14, about the vertex $n$, with positive angular velocity $2\omega$, and project upon the $YY$ axis, we shall obtain a diagram like that of Fig. 17, except that the order of vector succession will be reversed, in accordance with the projected curves of Fig. 15.

But the rotating energy vector of Fig. 17 only projects the fluctuation of energy in the circuit. There is, in addition to this fluctuation of energy, a steady stream of energy delivered from the circuit, corresponding to the steady average power.
$O P_m$ of Figs. 13, 15 and 16. If we represent the steady energy stream by the straight line $O P_m$ in Fig. 18, which shows 8 myria-
jourles in 0.01 second, or at the rate of 8 megajoules per second, then the resultant oscillation of energy, by Figs. 12, 13, 14, 15,
and 16, is the sinusoidal curve $W W$, completing two cycles in
1/60th of a second. Superposing the fluctuation on the steady
stream, we obtain the broken wavy line $a, b, c, d$, of energy de-
ivered from the generator against time, or current phase, as
abscesses. It will be seen that, between 150 and 200 degrees
of phase, the energy flow halts, and slightly reverses. At this time,
and at corresponding times in successive energy cycles, the
energy ceases to flow from the generator to the circuit; but
ebbs back from the circuit to the generator.

The above state of energy affairs may be represented projec-
tively by imparting to the center of the uniformly rotating energy
vector $NL$ or $n l$ (Fig. 17), a uniform velocity of translation of
$P_m$ joules per second in the direction of the $Y$ axis. But this is
equivalent to mounting the energy vector on a wheel whose axis
is at $N$ (Fig. 19), the rotation being on the $X Y$ plane. The
tread radius of the wheel is made equal to $W_f$, to joules scale, the
flange radius of the wheel is made equal to $W$, to joules scale.
A point on the flange is then allowed to project orthogonally on
the $Y Y$ axis, as the wheel rolls on the $Y Y$ rail at the angular

![Fig. 17.—Rotative-vector energy diagram for current standard phase.](image-url)
velocity $2\omega$ radians per second. The path of the moving flange point $L$, on the $XY$ plane, will be an oblate trochoid as shown, and the projection of the point, at 24 successive equal time intervals during the motion, is indicated on the line $yy$. It will be seen that the trochoid forms a small receding loop at the end of its curve, and while the tracing point describes this loop, the energy projection-point undergoes a small recession. If the tracing plane $YZO$ should move uniformly in the direction $OZ$ during the motion, the trace could be made to correspond to the curve $a, b, c, d$, of Fig. 18.

If the circuit, instead of being inductively reactive with a power-factor of 0.936, should be non-inductive (with power-factor 1.0), the flange on the rolling wheel disappears. The

---

**Fig. 18.—Energy-time diagram of alternating-current generator.**
tracing point then lies on the tread of the wheel. As shown in Fig. 20, the tracing point describes a cycloid in the $XOY$ plane, and the recessional loops in the tracing path disappear. The energy delivered by the generator to the circuit then halts at each cycle of energy vector rotation, or turn of the wheel; but it does not reverse.

If, on the contrary, the impedance factor $Z/R$ of the circuit increases, either inductively or condensively, the flange radius bears the same proportion to the tread radius. The oblate trochoïd described, as in Fig. 21, by the tracing point on the flange, develops larger recessional loops to the left of the line $YY$, with more marked reversals of motion and energy in the projected point.

Analytically, if $W$ be the flange radius, and $W_j$ the tread radius of the rolling wheel, the distance of the projected point along $yy$ from the initial position at which $W$ lies parallel to the $-XX$ axis is:

$$w_t = W_j 2\omega t + W \sin 2\omega t$$

joule-scale cm.  \(22\)
or substituting for \( W_t \) and \( W \) their values \( P_1t/2\omega \) and \( P/2\omega \) respectively,

\[
\begin{align*}
w_t &= P_1t + \frac{P}{2\omega} \sin 2\omega t \\
\therefore \quad p_t &= \frac{d w_t}{dt} = P_1 + P \cos 2\omega t
\end{align*}
\]

as already deduced from Figs. 13 and 15.

Finally, if the circuit be purely reactive, or resistanceless, the tread radius becomes zero, and the wheel spins without rolling. The path of the tracing point on the flange is then a pure circle. Its projection on the \( YY \) axis is a simple harmonic motion, and on the \( YOZ \) plane, a pure sinusoid.

Consequently, with suitable convention as to direction and sign of translatory motion, the energy given to an alternating current circuit may be represented by the vertically falling shadow, on the horizontal supporting rail, of a point on the flange of a uniformly rolling wheel. The rate of motion of this shadow will be
the power given to the circuit. The average power will be represented by the uniform speed of the axle. The instantaneous power by the instantaneous velocity of the shadow. If a second wheel be geared with the axle so as to rotate in the same direction with half the angular velocity, suitably selected radii on the latter wheel will represent the p. d. and current in the circuit, the instantaneous projections of these radii being measured on a vertical coordinate axis, and not on the horizontal rail.

Conformity of the Algebra of the Alternating-circuit and Continuous-current Circuits

Finally, it should be pointed out that just as, in regard to impedances, admittances, currents, and e.m.fs., the algebra of the single-frequency alternating-current circuit is the same as the algebra of the continuous-current circuit, the former dealing with complex quantities while the latter deals with real quantities; so the algebra of both circuits is the same, or at least may be regarded as the same, in dealing with power. For the power product of an e.m.f. $E/\theta$ and a current $I/\theta$, or of an $E/\theta$ and a current $I/0$, is $EI/\theta = IE/\theta$ watts and not $EI \cos \theta$ watts. The average externally delivered power, as well as the average instantaneous power, is however $EI \cos \theta$ watts and the maximum cyclic internally-reactive power is $EI \sin \theta$ watts.

Summary of Conclusions

The algebra and geometry of vector alternating-current technology, as developed in text books, are, at present, in a state of great and unnecessary confusion as to direction of rotation.

The confusion has existed for more than twenty years, and is not confined to any one country or language.

Calling that representation "direct" which denotes a leading current as leading, in the order of positive rotation, some two thirds of the alternating-current text-books use direct representation, and the remaining one third inverse representation.

The existing dissension relates to conventions and not to facts.

The directions of rotation and representation should be standardized by mutual international agreement.

1. This law was first announced by the writer, restricted however to impedances and admittances, in the paper on "Impedance", Transactions A.I.E.E., April 1893, Vol. 10, pp. 175–232. The law was speedily extended by Dr. C. P. Steinmetz to cover currents and e.m.fs.
The terms "real power" and "wattless power" are inaccurate and misleading.

It is readily possible to compute and discuss the power in an alternating-current circuit as a stationary-vector quantity, without reference to the double frequency of rotative-vector power.

The power developed by any single-frequency alternating e.m.f. $E$ root-mean-square volts on a co-frequent current $I/\pm \theta$ root-mean-square amperes, is algebraically $EI/\pm \theta$ watts. The externally liberated power is $EI \cos \theta$ watts, and is usually the principal consideration in power-transmission systems.

In any alternating-current circuit, or portion of the same, there are four non-rotating vectors $Z$, $E$, $P$, $W$ to standard current phase, and also four $Y$, $I$, $P$, $W$, to standard potential-difference phase, all connected by ordinary vector arithmetic, and not involving double-frequency products.

The energy in a single-frequency alternating-current circuit follows the projection, upon the rail, of a flange-point on a wheel rolling along the rail with uniform angular velocity. The path of the flange-point is an oblate trochoid for reactive circuits—but is a cycloid for a non-reactive circuit.

The algebra of alternating currents may be regarded as the same as the algebra of continuous currents, for power as well as for other quantities; so that any formula relating to direct-current circuits is also a formula relating to single-frequency alternating-current circuits, when complex numbers are substituted for real numbers.

**Notation Employed**

- $a$, $b$, Vector lengths (cm.).
- $\alpha$, $\beta$, $\theta$, Vector angles, or phase angles (radians or degrees).
- $B$, Susceptance of a circuit, as a whole, or beyond a pair of points in the same (mhos).
- $c$, Capacity of a condenser (farads).
- $E_0$, $E_E$, $E_k$, Maximum cyclic, virtual or root-mean-square, effective, and reactive e.m.f. in a circuit, or potential-difference at a pair of points in the same (volts).
- $e_t$, Instantaneous voltage or P. D. in a circuit at time $t$ (volts).
- $e = 2.71828\ldots$ (numeric).
- $G$, Conductance of a circuit as a whole, of beyond a pair of points in the same (mhos).
$I_o, I, I_t, I_k$ Maximum cyclic, virtual or root-mean-square, effective, and reactive current strength in a circuit (amperes).

$i_t$ Instantaneous current strength in a circuit (amperes).

$j = \sqrt{-1}$ (quadrantal operator).

$L$ Inductance in a circuit (henrys).

$l$ Polar radius-vector of a complex number.

$n$ Frequency of alternation (cycles per second).

$\omega$ Angular velocity of a rotating vector, or of an alternating quantity, (radians per second, or degrees per second).

$P, P_f, P_k$, Resultant, effective, and reactive maximum cyclic power in a circuit (watts).

$P_m$ Average power in a circuit (watts).

$p_t$ Instantaneous power in a circuit at time $t$ (watts).

$\pi = 3.14159 \ldots$ (numeric).

$R, r$, Resistance in a circuit, or conductor (ohms).

$T$, A constant time interval (seconds).

$t$ Elapsed time interval (seconds).

$W, W_f, W_k$, Resultant, effective and reactive maximum cyclic energy (joules).

$w_t$ Instantaneous energy in a circuit (joules).

$X, x$, Reactance in a circuit (ohms).

$Y$, Admittance, or resultant conductance, in a circuit (mhos).

$Z, z$, Impedance, or resultant resistance in a circuit (ohms).

$X X, Y Y, Z Z$, Rectangular coordinate axes.